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CHARACTERIZING WHEN $R[X]$ IS INTEGRALLY CLOSED, II

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For an integrally closed reduced ring R it is not always the case that the polynomial ring $R[X]$ is integrally closed. In this paper, the question of when $R[X]$ is integrally closed is shown to be related to when R is integrally closed in the ring of finite fractions of R, $Q_0(R)$. In the main theorem it is shown that if R' is the integral closure of R in $Q_0(R)$, then $R'[X]$ is the integral closure of $R[X]$ in $T(R[X])$, the total quotient ring of $R[X]$. This result is then used to characterize when $R[X]$ satisfies any of the weaker integrality properties of being n-root closed, root closed, or seminormal.

0. Preliminaries

In what follows, R is assumed to be a commutative ring with unity and no nonzero nilpotents; i.e., R is a reduced ring. We denote by $T(R)$ the total quotient ring of R and when we say that R is integrally closed we mean that R is integrally closed in $T(R)$.

A weaker integrality property is that of seminormality. A ring R is said to be seminormal in a ring T if $R \subset T$ and for each $t \in T$, whenever t^2 and t^3 are in R, then t is in R. As with being integrally closed, R is said to be seminormal if R is seminormal in $T(R)$.

In [6, Theorem 4] we gave necessary and sufficient conditions in order that the polynomial ring $R[X]$ be integrally closed. We will give another characterization here and use it to determine when $R[X]$ satisfies any of the weaker integrality properties of being n-root closed, root closed, or seminormal.

Two notions which will play a prominent role in our work are those of a dense ideal and the content of a polynomial. As in [5], we say that an ideal I is dense if $rI=(0)$ implies $r=0$. For a polynomial $g \in R[X]$, the content of g is the ideal $c(g)$ of R generated by the coefficients of g. A well-known theorem of McCoy states that $g \in R[X]$ is a zero divisor if and only if there is a nonzero $r \in R$ such that $rg=0$. It

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is possible that the content of g contains only zero divisors of R yet g is not a zero divisor of $R[X]$; in other words, $c(g)$ can be finitely generated dense ideal which contains only zero divisors.

Throughout the paper g_j denotes the coefficient on X^j for $g(X) \in R[X]$. Any other notation or terminology is standard as in [1] or [3].

1. Integrality properties of $R[X]$

In [6] we established necessary and sufficient conditions for $R[X]$ to be integrally closed. These conditions were stated in the negative; e.g., $R[X]$ is not integrally closed if and only if there is a quotient of polynomials $f/g \in T(R[X]) \setminus R[X]$ which is integral over R [6, Theorem 4]. From the proof of this statement, we see that multiplication by f/g defines an R-module homomorphism from the content of g to R. In particular, $(f/g)g_j = f_j$ for each pair of coefficients f_j and g_j of f and g respectively. Thus we are led to consider the so called *ring of finite fractions* of R which we denote by $Q_o(R)$. One way to construct the ring $Q_o(R)$ is to repeat the construction given in Lambek's book [6, pp. 36–46] for the complete ring of quotients $Q(R)$ replacing the phrase "dense ideal" with "finitely generated dense ideal". We sketch the construction here.

Let J_1 and J_2 be finitely generated dense ideals of R and let $f_1 \in \mathrm{Hom}_R(J_1, R)$, $f_2 \in \mathrm{Hom}_R(J_2, R)$. Then $J_1 J_2$ is a finitely generated dense ideal of R so that we may define $f_1 + f_2$ and $f_1 f_2$ as homomorphisms on $J_1 J_2$. Define f_1 and f_2 to be equivalent if they agree on a dense ideal J of R. From [5, Lemma 1, p. 38] we see that f_1 and f_2 agree on a dense ideal J if and only if they agree on $J_1 \cap J_2$ and hence if and only if they agree on $J_1 J_2$. The elements of $Q_o(R)$ are the equivalence classes of the homomorphisms.

As with $Q(R)$, $Q_o(R)$ contains both R and $T(R)$ as subrings. Moreover, $Q_o(R) \subset Q(R)$ and so $T(R[X]) \subset T(Q_o(R)[X])$ since $T(R[X]) \subset T(Q(R)[X])$.

As we have assumed that R is reduced, $Q(R)$ is von Neumann regular. Hence, $Q(R)[X]$ is integrally closed (see for example [4, p. 224]). Thus if we let S be the integral closure of R in $Q(R)$, then $S[X]$ is integrally closed in $T(Q(R)[X])$ since it is integrally closed in $Q(R)[X]$. Moreover, as $T(R[X]) \subset T(Q(R)[X])$ we have that the integral closure of $R[X]$ in $T(R[X])$ is contained in $S[X]$. As our main theorem we will show that the integral closure of $R[X]$ in $T(R[X])$ can be identified with the ring $R'[X]$ where R' is the integral closure of R in $Q_o(R)$. We use this result to characterize when $R[X]$ satisfies the weaker integrality properties of being n-root closed, root closed, and semi-normal.

Before presenting the main theorem we need a pair of lemmas.

Lemma 1. *Let $s \in Q_o(R)$. Then there exist polynomials f and g in $R[X]$ with $c(g)$ dense in R such that $s \in \mathrm{Hom}_R(c(g), R)$ and for each j, $s(g_j) = f_j$. In particular, $Q_o(R)$ can be considered as a subring of $T(R[X])$.*

Proof. As $s \in Q_\circ(R)$ there exists a finitely generated dense ideal $J = (g_0, g_1, \ldots, g_n)$ such that $s \in \mathrm{Hom}(J, R)$. For each $j = 0, 1, \ldots, n$ set $s(g_j) = f_j$, $f = \sum f_j X^j$ and $g = \sum g_j X^j$. Then for each i and j, $f_i g_j = f_j g_i$ since $s(g_i g_j) = g_i s(g_j) = g_j s(g_i)$. Hence, $(f/g) g_j = f_j$ for each $j = 0, 1, \ldots, n$ and we can consider s as an element of $T(R[X])$. □

Lemma 2. *Let $s \in Q(R)$. If $sJ \subset Q_\circ(R)$ for some finitely generated dense ideal J of R, then $s \in Q_\circ(R)$.*

Proof. Let $J = (a_0, a_1, \ldots, a_n)$ be a dense ideal of R such that $sa_i \in Q_\circ(R)$ for each $i = 0, 1, \ldots, n$. By Lemma 1 there exist polynomials F_i and G_i in $R[X]$ such that $sa_i = F_i / G_i$.

Let t be the maximum degree of the G_i's and define polynomials F and G by

$$F(X) = sa_0 G_0 + sa_1 G_1 X^{t+1} + sa_2 G_2 X^{2t+2} + \cdots + sa_n G_n X^{nt+n}$$

and

$$G(X) = a_0 G_0 + a_1 G_1 X^{t+1} + a_2 G_2 X^{2t+2} + \cdots + a_n G_n X^{nt+n}.$$

That the content of G is dense follows from McCoy's Theorem; if $r \in R$ is such that $rc(G) = (0)$, then $ra_i G_i = 0$ implies $ra_i = 0$ for each i. As J is dense, $r = 0$. Thus $c(G)$ is dense and $s = F/G \in \mathrm{Hom}(c(G), R)$. □

Remark. While the result above is not really new, (see for example [8, pp. 151, 197 & 201]), we have not been able to find in the literature, a previous proof involving quotients of polynomials. Hence we have given the above proof as a way to further illustrate the relation between $Q_\circ(R)$ and $T(R[X])$.

Theorem 3. *Let R' be the integral closure of the reduced ring R in $Q_\circ(R)$. Then $R'[X]$ is the integral closure of $R[X]$ in $T(R[X])$.*

Proof. Let $f/g \in T(R[X])$ be integral over $R[X]$. Then from the discussion above we have that $f/g = s(X) \in S[X]$. By way of contradiction assume that $f/g \notin Q_\circ(R)$. Then we may assume that $s(X)$ has minimal degree for all such quotients.

Write $s(X) = s_k X^k + s_{k-1} X^{k-1} + \cdots + s_0$ and $g(x) = g_m X^m + g_{m-1} X^{m-1} + \cdots + g$. Then mimicking the proof of [6, Theorem 4], we see that $s_k \in S \setminus Q_\circ(R)$ and that $s_k g_j \in Q_\circ(R)$ for each j. As the content of g is dense, we have that, contrary to assumption, $s_k \in Q_\circ(R)$. Hence, $f/g \in Q_\circ(R)[X]$. As $R'[X]$ is the integral closure of $R[X]$ in $Q_\circ(R)[X]$, $R'[X]$ is the integral closure of $R[X]$ in $T(R[X])$. □

Corollary 4. *For a reduced ring R, $R[X]$ is integrally closed if and only if R is integrally closed in $Q_\circ(R)$.* □

We can also use Theorem 3 to characterize when $R[X]$ satisfies any of the weaker integrality properties of being n-root closed, root closed and seminormal. Note that according to [7, Example 1.4] it is possible for $R[X]$ to be root closed without being integrally closed even if $R=T(R)$.

Corollary 5. *Let R be a reduced ring. Then $R[X]$ is n-root closed if and only if R is n-root closed in $Q_o(R)$.*

Proof. If R is not n-root closed in $Q_o(R)$, then there exists an $s\in Q_o(R)\setminus R$ such that $s^n\in R$. As s can be written as a quotient of polynomials, $s\in T(R[X])$ and so $R[X]$ is not n-root closed.

On the other hand, if R is n-root closed in $Q_o(R)$, then R is n-root closed in R', the integral closure of R in $Q_o(R)$. Hence by [2, Theorem 1], $R[X]$ is n-root closed in $R'[X]$. As $R'[X]$ is the integral closure of $R[X]$ in $T(R[X])$, $R[X]$ is n-root closed. □

The same proof holds if we replace n-root closed by either root closed or seminormal, the latter by way of [2, Theorem 2]. Hence we have a final corollary.

Corollary 6. *Let R be a reduced ring. Then $R[X]$ is root closed (seminormal) if and only if R is root closed (seminormal) in $Q_o(R)$.* □

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